34th Indian National Mathematical Olympiad - 2019

Time : 4 hours January 20, 2019

Instructions:

Calculators (in any form) and protractors are not allowed.

Rulers and compasses are allowed.

All questions carry equal marks. Maximum marks : 102.

Answer all the questions.

Answer to each question should start on a new page. Clearly indicate the question number.

1. Let ABC be a triangle with $\angle BAC > 90^\circ$. Let D be a point on the segment BC and E be a point on the line AD such that AB is tangent to the circumcircle of triangle ACD at A and BE is perpendicular to $AD.$ Given that $CA = CD$ and $AE = CE,$ determine $\angle BCA$ in degrees.

2. Let $A_1B_1C_1D_1E_1$ be a regular pentagon. For $2 \leq n \leq 11,$ let $A_nB_nC_nD_nE_n$ be the pentagon whose vertices are the midpoints of the sides of $A_{n-1}B_{n-1}C_{n-1}D_{n-1}E_{n-1}.$ All the 5 vertices of each of the 11 pentagons are arbitrarily coloured red or blue. Prove that four points among these 55 points have the same colour and form the vertices of a cyclic quadrilateral.

3. Let m, n be distinct positive integers. Prove that
$gcd(m,n) + gcd(m+1,n+1) + gcd(m+2,n+2) \leq 2|m-n| + 1.$
Further, determine when equality holds.

4. Let n and M be positive integers such that $M > n^{n-1}.$ Proove that there are n distinct primes $p_1, p_2, p_3, ...,p_n$ such that p_j divides $M + j$ for $1 \leq j \leq n.$

5. Let AB be a diameter of a circle Γ and let C be a point on Γ different from A and $B.$ Let D be the foot of perpendicular from G on to $AB.$ Let K be a point of the segment CD such that AC is equal to the semiperimeter of the triangle $ADL.$ Show that the excircle of triangle ADK opposite A is tangent to $\Gamma.$

6. Let f be a function defined from the set $\{(x,y) : x, y \,\text{real}, xy \neq 0\}$ to the set of all positive real numbers such that
(i) $f(xy,z) = f(x,z)f(y,z),$ for all $x, y \neq 0;$
(ii) $f(x, 1-x) = 1,$ for all $x \neq 0, 1.$
Prove that
(a) $f(x,x) = f(x,-x) = 1,$ for all $x \neq 0;$
(b) $f(x,y)f(y,x) = 1,$ for all $x, y \neq 0.$

Space for rough use