Rao IIT Academy
Symbol of Excellence and Perfection
JEE | MEDICAL-UG | BOARDS | KVPY | NTSE | OLYMPIADS

Date: 13-05-2018	I.S.I - 2018	Max. Marks : 120 Max. Time : 2 hrs

❖ ISI - 2018_OFFICAL TEST_MATHEMATICS_SUBJECTIVE ❖

Notations : In the following, $\mathbb{N}=\{1,2,3,....\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers.

Q.1 **Find all pairs (*x*, *y*) with *x*, *y* real, satisfying the equations :**

$$\sin\left(\frac{x+y}{2}\right)=0, |x|+|y|=1$$

Sol. $\sin\left(\frac{x+y}{2}\right)=0 \Leftrightarrow \frac{x+y}{2}=n\pi$

$\Leftrightarrow x+y=2n\pi \qquad n\in I$

as $|x+y|\le|x|+|y|$

$\Rightarrow |2n\pi|\le 1 \quad n\in I$

$\Leftrightarrow n=0$

$\Rightarrow x+y=0$ i.e. $y=-x$

$\Rightarrow |x|+|x|=1 \Leftrightarrow |x|=\frac{1}{2} \Rightarrow x=\pm\frac{1}{2}$

Hence $(x,y)=\left(\frac{1}{2},-\frac{1}{2}\right)$ or $\left(-\frac{1}{2},\frac{1}{2}\right)$ i.e. 2 solutions

Q.2 **Suppose that *PQ* and *RS* are two chords of a circle intersecting at a point *O*. It is given that *PO* = 3 *cm* and *SO* = 4 *cm*. Moreover, the area of the triangle *POR* is 7 cm^2. Find the area of the triangle *QOS*.**

Sol. $\Delta POR \sim \Delta SOQ$

$$\Rightarrow \frac{A(\Delta POR)}{A(\Delta SOQ)}=\left(\frac{PO}{SO}\right)^2=\frac{9}{16}$$

$$\Rightarrow A(\Delta SOQ)=\frac{16}{9}\cdot 7=\frac{112}{9}$$

Q.3 **Let** $f:\mathbb{R}\to\mathbb{R}$ **be a continuous function such that for all** $x\in\mathbb{R}$ **and for all** $t\geq 0$,

$$f(x)=f\left(e^{t}x\right)$$

Show that *f* is a constant function.

Sol. Put $e^{t}x=y \Rightarrow x=e^{-t}y$

$\Rightarrow f\left(e^{-t}y\right)=f(y)\forall t$

Take $t\to\infty$

$\Rightarrow \lim\limits_{t\to\infty} f(y)=\lim\limits_{t\to\infty} f\left(e^{-t}y\right)$

$\Rightarrow f(y)=\lim\limits_{h\to 0} f(h)$

where $h=e^{-t}y$ and $h\to 0$ as $t\to\infty$

but as $f(x)$ is a continuous function, it is continuous at $x=0$ i.e. $\lim\limits_{h\to 0} f(h)=f(0)$

$\Rightarrow f(y)=f(0)\forall y \Rightarrow f$ is constant function

Q.4 **Let** $f:(0,\infty)\to\mathbb{R}$ **be a continuous function such that for all** $x\in(0,\infty)$,

$$f(2x)=f(x)$$

Show that the function *g* defined by the equation

$$g(x)=\int_{x}^{2x} f(t)\frac{dt}{t} \quad \text{for} \quad x>0$$

is a constant function.

Sol. As $f(x)$ is continuous function $\dfrac{f(t)}{t}$ is a continuous function for $t>0$, hence by fundamental theorem of calculus, the function $g(x)=\int_{x}^{2x}\dfrac{f(t)}{t}dt$ is a differentiable function $\forall x>0$ and derivative is given by

$$g'(x)=\frac{f(2x)\cdot 2}{2x}-\frac{f(x)}{x}=\frac{f(2x)-f(x)}{x}=0$$

$\Rightarrow g(x)$ is constant function $\forall x>0$

(It is a standard proof that if $g'(x)=0,$ throughout a continuos domain, $g(x)$ is constant)

Q.5 **Let** $f:\mathbb{R}\to\mathbb{R}$ **be a differentiable function such that its derivative** f' **is a continuous function. Moreover, assume that for all** $x\in R$,

$$0\le |f'(x)|\le\frac{1}{2}$$

Define a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ **by :**

$$a_1=1,$$

$$a_{n+1}=f(a_n)\text{ for all } n\in\mathbb{N}$$

Prove that there exists a positive real number *M* such that for all $n\in\mathbb{N}$,

$$|a_n|\le M$$

Sol. As $-\frac{1}{2}\le f'(x)\le\frac{1}{2}$

Integrating from 0 to x, we get

$-\frac{x}{2}\le f(x)-f(0)\le\frac{x}{2}$ take $f(0)=c$

$\Rightarrow c-\frac{x}{2}\le f(x)\le c+\frac{x}{2}$

Put $x=a_n$

$\Rightarrow c-\frac{a_n}{2}\le f(a_n)=a_{n+1}\le c+\frac{a_n}{2}\ \forall n\in\mathbb{N}$

$\Rightarrow c-\frac{a_{n-1}}{2}\le a_n\le c+\frac{a_{n-1}}{2}$ (1)

Now

$$\Rightarrow a_{n+1}\le c+\frac{a_n}{2}\le c+\frac{c}{2}+\frac{a_{n-1}}{2^2}\le\ldots\le \overbrace{c+\frac{c}{2}+\ldots+\frac{c}{2^{n-1}}+\frac{a_1}{2^n}}^{\text{easy to prove by induction}}$$

$$\le|c|\left|1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}\right|+\frac{1}{2^n}\le|c|\left|1+\frac{1}{2}+\ldots\infty\right|+\frac{1}{2^n}\le 2|c|+1$$

Hence a_n is bounded above ...(2)

Similarly

$a_{n-1}\le c+\frac{a_{n-2}}{2}$

$\Leftrightarrow -c-\frac{a_{n-2}}{2}\le -a_{n-1}$

$\Leftrightarrow c-\frac{c}{2}-\frac{a_{n-2}}{2^2}\le c-\frac{a_{n-1}}{2}\le a_n$ From (1)

$\vdots$

$$c-\frac{c}{2}-\frac{c}{2^2}-....-\frac{c}{2^{n-2}}-\frac{a_1}{2^{n-1}}\le a_n$$

$$\Rightarrow -\left|c-\frac{c}{2}-\frac{c}{2^2}-...-\frac{c}{2^{n-2}}\right|-\frac{1}{2^{n-1}}\le c-\frac{c}{2}-\frac{c}{2^2}-....-\frac{c}{2^{n-2}}-\frac{a_1}{2^{n-1}}\le a_n \quad \text{Using } -|x|\le x$$

$$\Rightarrow -|c|\overbrace{\left|1-\frac{1}{2}-\frac{1}{2^2}-....-\frac{1}{2^{n-2}}\right|}^{=\frac{1}{2^{n-2}}}-\frac{1}{2^{n-1}}\le a_n$$

$$\Rightarrow -|c|\frac{1}{2^{n-2}}-\frac{1}{2^{n-1}}\le a_n$$

$$\Rightarrow -\frac{(2|c|+1)}{2^{n-1}}\le a_n$$

$$\Rightarrow -(2|c|+1)\le -\frac{(2|c|+1)}{2^{n-1}}\le a_n \quad \Rightarrow a_n \text{ is bounded below} \qquad ...(3)$$

(2) and (3) $\Rightarrow a_n$ is bounded i.e. these exists

$m=2|c|+1$ for which

$|a_n|\le m=2|c|+1$

Q.6 Let $a\ge b\ge c>0$ be real numbers such that for all $n\in\mathbb{N}$, there exist triangles of side lengths a^n, b^n, c^n. Prove that the triangles are isosceles.

Sol. Assume the triangle not isoscales $\Rightarrow a>b>c$

since a^n, b^n, c^n form triangle

$$\Rightarrow b^n+c^n>a^n$$

$$\Rightarrow \left(\frac{b}{a}\right)^n+\left(\frac{c}{a}\right)^n>1 \quad \forall n$$

Now take $n\to\infty$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{b}{a}\right)^n+\left(\frac{c}{a}\right)^n\ge 1 \text{ but as } \lim_{n\to\infty}x^n=0 \quad \forall |x|<1$$

$\Rightarrow 0\ge 1$ which is a contraction

$\Rightarrow$ The triangle is isoscales. Also more precisely $a=b\ge c$ as $a>b=c$ would violate the above triangle inequality as well. (Here we've used $\lim_{n\to\infty}\left(\frac{b}{a}\right)^n=1$ if $b=a$ to justify isoscales triangle)

Q.7 **Let** $a, b, c \in \mathbb{N}$ **be such that**

$$a^2 + b^2 = c^2 \text{ and } c - b = 1$$

Prove that,

(i) ***a* is odd,**

(ii) ***b* is divisible by 4,**

(iii) $a^b + b^a$ **is divisible by *c***

Sol. (i) $a^2 = c^2 - b^2 = c + b = 2b + 1 \equiv 1 \pmod 2$

$\Rightarrow a \neq 0 \pmod 2$ as $a^2 \equiv 1 \pmod 2 \Rightarrow a$ is odd

(ii) Take $a = 2k + 1$

$\Rightarrow b = \dfrac{a^2 - 1}{2} = 2k(k+1) \equiv 0 \pmod 4$ as $k(k+1)$ is even hence b is multiple of 4

(iii) $a^2 \equiv c + b = 2c - 1 \equiv -1 \pmod{\text{c}}$

$\Rightarrow a^b = a^{2k(k+1)} \equiv (-1)^{k(k+1)} = 1 \pmod c$ as $k(k+1)$ is even

As $b = c - 1 \equiv -1 \pmod c$

$\Rightarrow b^a = b^{2k+1} \equiv (-1)^{2k+1} \equiv -1 \pmod c$

$\Rightarrow a^b + b^a \equiv 1 - 1 \equiv 0 \pmod c$

i.e. $a^b + b^a$ is divisible by c.

Q.8 **Let** $n \geq 3$**. Let** $A = ((a_{ij}))_{1 \leq i, j \leq n}$ **be an** $n \times n$ **matrix such that** $a_{ij} \in \{1, -1\}$ **for all** $1 \leq i, j \leq n$**. Suppose that**

$$a_{k1} = 1 \text{ for all } 1 \leq k \leq n \text{ and}$$

$$\sum_{k=1}^{n} a_{ki} a_{kj} = 0 \text{ for all } i \neq j$$

Show that *n* is a multiple of 4.

Sol. In $\sum_{1}^{n} a_{ki} a_{kj} = 0 \qquad (i \neq j)$

Put $i = 1$ (Assume $j \neq 1$ now onwards)

$\Rightarrow \sum_{1}^{n} a_{k1} a_{kj} = 0$

As $a_{k1} = 1$

$\Rightarrow a_{1j} + a_{2j} + a_{3j} + ... + a_{nj} = 0$

i.e. sum of elements in $j^{th} (\neq 1)$ column is zero

As $a_{kj} = 1$ or -1, assume m out of n entries in j^{th} column are -1, hence the rest $n-m$ entries will be 1 and as their sum is zero, we get

$$\underbrace{(-1)+(-1)+\ldots+(-1)}_{m \text{ times}}+\underbrace{1+1+\ldots+1}_{n-m \text{ times}}=0$$

$\Leftrightarrow -m+n-m=0 \Rightarrow n=2m$ i.e. n is atleast even.

Hence no. of 1 and -1 should be equal to in any $j^{th}(\neq 1)$ column

Now in $\sum_{1}^{n} a_{ki}\, a_{kj} = 0$

i.e. $a_{1i}\, a_{1j} + a_{2i}\, a_{2j} + \ldots + a_{ni}\, a_{nj} = 0 \quad (i \neq j)$

Some terms are "$(-1)(-1)$", some are "$(-1)(1)$", some are "$(1)(-1)$" and the rest are "$(1)(1)$".

Let there be r number of "$(-1)(-1)$" terms

$\Rightarrow$ there will be $m-r$ "$(-1)(1)$" terms as total m terms in the i^{th} column are -1 for a_{ki}

$\Rightarrow$ $m-r$ terms will be "$(1)(-1)$" and hence r terms will be "$(1)(1)$"

As the sum is zero

$\Rightarrow r(-1)(-1)+(m-r)(-1)(1)+(m-r)(1)(-1)+r(1)(1)=0$

$\Rightarrow 2r-2(m-r)=0$

$\Rightarrow m=2r \Rightarrow n=4r$ hence a is multiple of 4